SOLUTIONS

UBC Math 104/184 Exam (December 2011)

1. (a) $\lim\limits_{x\to 4}\dfrac{x^2-3x-4}{\sqrt{x}-2} = \lim\limits_{x\to 4}\dfrac{(x-4)(x+1)}{\sqrt{x}-2}\cdot\dfrac{\sqrt{x}+2}{\sqrt{x}+2} = \lim\limits_{x\to 4}\dfrac{(x-4)(x+1)(\sqrt{x}+2)}{x-4}$

$= \lim\limits_{x\to 4}(x+1)(\sqrt{x}+2) = (4+1)(\sqrt{4}+2) = 5\cdot 4 = 20$

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1. (b) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(2x^2-\ln x) = 2\cdot 1^2-\ln 1 = 2-0 = 2$;

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(x^3-4ax) = 1^3-4a\cdot 1 = 1-4a$.

In order for $f(x)$ to be continuous at $x=1$, $\lim\limits_{x\to 1} f(x)$ must exist, which can only happen if

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x)$, i.e. $2 = 1-4a$, or $4a=-1$. So $a=-\frac{1}{4}$.

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1. (c) $f'(x) = \dfrac{(x^4+7)\cdot\frac{d}{dx}[(x^2+3\sin^2 x)e^{x^2}]-[(x^2+3\sin^2 x)e^{x^2}]\cdot\frac{d}{dx}(x^4+7)}{(x^4+7)^2}$

$= \dfrac{(x^4+7)\cdot[(x^2+3\sin^2 x)\cdot\frac{d}{dx}(e^{x^2})+e^{x^2}\cdot\frac{d}{dx}(x^2+3\sin^2 x)]-[(x^2+3\sin^2 x)e^{x^2}]\cdot 4x^3}{(x^4+7)^2}$

$= \dfrac{(x^4+7)[(x^2+3\sin^2 x)\cdot(e^{x^2}\cdot 2x)+e^{x^2}(2x+6\sin x\cdot\cos x)]-4x^3(x^2+3\sin^2 x)e^{x^2}}{(x^4+7)^2}$

$= \dfrac{e^{x^2}\{(x^4+7)[2x(x^2+3\sin^2 x)+(2x+6\sin x\cos x)]-4x^3(x^2+3\sin^2 x)\}}{(x^4+7)^2}$

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1. (d) When $x=0$, $y=f(0)=\dfrac{0+3}{0+1}=3$, so the point on the curve is $(x,y)=(0,3)$.

Since $f'(x) = \dfrac{(2x+1)\cdot 1-(x+3)\cdot 2}{(2x+1)^2} = \dfrac{(2x+1)-(2x+6)}{(2x+1)^2} = -\dfrac{5}{(2x+1)^2}$,

the slope of the tangent line to the curve at $(0,3)$ is $m=f'(0)=-\dfrac{5}{(0+1)^2}=-5$.

Since $(0,3)$ is the y-intercept, then $b=3$, and the equation of the tangent line is

$y=mx+b=-5x+3$.

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1. (e) Since we cannot take the square root of a negative number, the function $f(x)$ is only defined if $2-x^2 \geq 0$, i.e. $x^2 \leq 2$ or $-\sqrt{2} \leq x \leq \sqrt{2}$. So the domain of the function $f(x)$ is $[-\sqrt{2}, -\sqrt{2}]$. The derivative of $f(x)$ is

$$f'(x) = x \cdot \frac{d}{dx}(2-x^2)^{1/2} + (2-x^2)^{1/2} \cdot \frac{d}{dx}(x)$$

$$= x \cdot \left[\frac{1}{2}(2-x^2)^{-1/2} \cdot (-2x)\right] + (2-x^2)^{1/2} \cdot 1 = \frac{-x^2}{\sqrt{2-x^2}} + \sqrt{2-x^2}$$

$$= \frac{-x^2}{\sqrt{2-x^2}} + \sqrt{2-x^2} \cdot \frac{\sqrt{2-x^2}}{\sqrt{2-x^2}} = \frac{-x^2}{\sqrt{2-x^2}} + \frac{2-x^2}{\sqrt{2-x^2}} = \frac{2-2x^2}{\sqrt{2-x^2}}.$$

At a critical point, $f'(x)=0$ so $2-2x^2 = 2(1-x^2)=0$. Therefore $x^2=1$ so $x=\pm 1$. So there are two critical points ($x=1$ and $x=-1$) and two endpoints ($x=\sqrt{2}$ and $x=-\sqrt{2}$).

At the critical point $x=1$, $y=f(1)=1$; at the critical point $x=-1$, $y=f(-1)=-1$.

At the endpoint $x=\sqrt{2}$, $y=f(\sqrt{2})=0$; at the endpoint $x=-\sqrt{2}$, $y=f(-\sqrt{2})=0$.

Thus the absolute maximum of $f(x)$ occurs when $x=1$.

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1. (f) Plug in $x=1$ to get

$$f'(1) = 1 \cdot [f(1)]^2 + 1^2 = 1 \cdot 2^2 + 1 = 5$$

Differentiating both sides of the original equation with respect to *x* gives

$$\frac{d}{dx}[f'(x)] = \frac{d}{dx}\left\{x[f(x)]^2 + x^2\right\},$$

or

$$f''(x) = \left(x \cdot \frac{d}{dx}[f(x)]^2 + [f(x)]^2 \cdot \frac{d}{dx}(x)\right) + 2x$$

$$= x \cdot 2f(x)f'(x) + [f(x)]^2 \cdot 1 + 2x = 2xf(x)f'(x) + [f(x)]^2 + 2x.$$

So

$$f''(1) = 2f(1)f'(1) + [f(1)]^2 + 2 = 2 \cdot 2 \cdot 5 + 2^2 + 2 = 26.$$

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1. (g) Notice that $f(x) = x^2 - \ln x$ is only defined for $x > 0$. Differentiate to get

$$f'(x) = 2x - \frac{1}{x} = \frac{2x^2-1}{x}.$$

$f(x)$ is increasing when $f'(x) > 0$; that is $2x^2 - 1 > 0$ (since $x > 0$). So $2x^2 > 1$ or $x^2 > \frac{1}{2}$ or $x > \sqrt{\frac{1}{2}}$.

So $f(x)$ is increasing on the interval $(\sqrt{\frac{1}{2}}, \infty)$.

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1. (h) Let $A(t)$ be the value of the stamp collection (in millions of dollars). Then $A(t)=Pe^{rt}$ where $P=10$ and $r=12\%=0.12$. So $A(t)=10e^{0.12t}$. When the value of the collection has tripled,

$$A(t)=10e^{0.12t}=30 \quad\Rightarrow\quad e^{0.12t}=3 \quad\Rightarrow\quad 0.12t=\ln 3 \quad\Rightarrow\quad t=\frac{\ln 3}{0.12}\approx 9.16\,.$$

It will take approximately 9.16 years for the stamp collection to triple in value.

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1. (i) $\dfrac{d}{dx}(xy^2+2xy) = \dfrac{d}{dx}(8) \quad\Rightarrow\quad \left(x\cdot\dfrac{d}{dx}(y^2)+y^2\cdot\dfrac{d}{dx}(x)\right)+2\left(x\cdot\dfrac{d}{dx}(y)+y\cdot\dfrac{d}{dx}(x)\right) = 0$

$$\Rightarrow\quad (x\cdot(2yy')+y^2\cdot 1)+2\,(x\cdot y'+y\cdot 1)=0$$

$$\Rightarrow\quad 2xyy'+y^2+2xy'+2y=0$$

Plugging in $(x,y)=(1,2)$ gives

$$(2\cdot 1\cdot 2)y'+2^2+(2\cdot 1)y'+2\cdot 2=0 \quad\Rightarrow\quad 4y'+4+2y'+4=0 \quad\Rightarrow\quad 6y'+8=0$$

so the slope of the tangent line is

$$m=y'\big|_{(1,2)}=-\tfrac{8}{6}=-\tfrac{4}{3}\,.$$

The equation of the tangent line is therefore

$$y-2=-\tfrac{4}{3}(x-1) \quad\text{or}\quad y=-\tfrac{4}{3}x+\tfrac{10}{3}\,.$$

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1. (j) The answer is **(A)**. If $f'(x)$ and $g'(x)$ are both continuous and increasing, then their derivatives will ALWAYS be positive, i.e. $f''(x)>0$ and $g''(x)>0$ for all *x*. Therefore

$$\frac{d^2}{dx^2}(f(x)+g(x))=f''(x)+g''(x)>0 \quad\text{for all } x,$$

so $f+g$ is concave up everywhere, and I is always true. However II may not always be true (for example if $f(x)=x^2$ and $g(x)=x^2$, then $f(x)-g(x)=0$). Neither is III always true (for example if $f(x)=e^x$ and $g(x)=e^{-x}$, then $f(x)\cdot g(x)=1$).

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1. (k) (D) need not necessarily be true. All of the rest must be true ((A) is true because of the Extreme Value Theorem, (B) because of the definition of function, (C) because of the Intermediate Value Theorem, and (E) since *g* is continuous on $[0,1]$). The answer is **(D)**.

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1. (l) The answer is **(B)**. (Because if $f'(a)$ exists, then f is differentiable at *a*, so f is continuous at *a*, and hence $\lim_{x\to a} f(x)=f(a)$.)

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1. (m) Let the two numbers be *x* and *y*. Then $xy = 50$ so $y = \dfrac{50}{x}$. The sum of the two numbers is then

$S = x + y = x + \dfrac{50}{x}$. So we need to minimize the function $f(x) = x + 50x^{-1}$.

Since $f'(x) = 1 - 50x^{-2} = 1 - \dfrac{50}{x^2} = 0$ at a critical point, then $1 = \dfrac{50}{x^2}$ or $x^2 = 50$ and therefore

$x = \sqrt{50} = 5\sqrt{2}$. Also $y = \dfrac{50}{x} = \dfrac{50}{\sqrt{50}} = \sqrt{50}$. Since $f''(x) = 0 + 100x^{-3} = \dfrac{100}{x^3} > 0$, this gives a minimum value for the sum. So the two numbers are $x = \sqrt{50} = 5\sqrt{2}$ and $y = \sqrt{50} = 5\sqrt{2}$.

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1. (n) $f'(x) \;=\; \frac{1}{3}x^{-2/3} \;=\; \dfrac{1}{3(\sqrt[3]{x})^2}$

The linear approximation is given by

$$f(x) \;\approx\; f(a) + f'(a)(x-a) \;=\; f(8) + f'(8)(x-8),$$

or $$\sqrt[3]{x} \;\approx\; \sqrt[3]{8} + \frac{1}{3(\sqrt[3]{8})^2}(x-8) \;=\; 2 + \tfrac{1}{12}(x-8).$$

Plugging in $x = 7.5$ gives

$$\sqrt[3]{7.5} \;\approx\; 2 + \tfrac{1}{12}(7.5-8) \;=\; 2 + \tfrac{1}{12}\cdot(-\tfrac{1}{2}) \;=\; 2 - \tfrac{1}{24} \;=\; \tfrac{47}{24}.$$

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2. (a) $f'(x) = 0$ if the numerator is zero, i.e.

$$f'(x) = 0 \quad\Rightarrow\quad (x^2-1)(x^2-6) = 0 \quad\Rightarrow\quad x^2 - 1 = 0 \;\text{ OR }\; x^2 - 6 = 0$$

$$\Rightarrow\quad x^2 = 1 \;\text{ OR }\; x^2 = 6 \quad\Rightarrow\quad x = \pm 1 \;\text{ OR }\; x = \pm\sqrt{6}$$

$f'(x)$ does not exist if the denominator is zero, i.e.

$$f'(x) \text{ does not exist} \quad\Rightarrow\quad (x^2-3)^2 = 0 \quad\Rightarrow\quad x^2 - 3 = 0 \quad\Rightarrow\quad x = \pm\sqrt{3}$$

So $f'(x) = 0$ if $x = \pm 1$ or $x = \pm\sqrt{6}$, and $f'(x)$ does not exist if $x = \pm\sqrt{3}$.

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2. (b) $f''(x) = 0$ if the numerator is zero, i.e.

$$f''(x) = 0 \quad\Rightarrow\quad 2x(x^2+9) = 0 \quad\Rightarrow\quad 2x = 0 \;\text{ OR }\; x^2 + 9 = 0 \quad\Rightarrow\quad x = 0 \;\text{ OR }\; x^2 = -9$$

Since x^2 cannot be -9, $f''(x) = 0$ only if $x = 0$.

$f''(x)$ does not exist if the denominator is zero, i.e.

$$f''(x) \text{ does not exist} \quad\Rightarrow\quad (x^2-3)^3 = 0 \quad\Rightarrow\quad x^2 - 3 = 0 \quad\Rightarrow\quad x = \pm\sqrt{3}$$

So $f''(x)=0$ if $x=0$, and $f''(x)$ does not exist if $x=\pm\sqrt{3}$.

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2. (c) The denominator of $f'(x)=\dfrac{(x^2-1)(x^2-6)}{(x^2-3)^2}$ is always positive so the sign of $f'(x)$ is determined by the sign of its numerator. The following table determines the intervals where $f'(x)$ is positive or negative.

Interval	x^2-1	x^2-6	$f'(x)=\dfrac{(x^2-1)(x^2-6)}{(x^2-3)^2}$	Increasing or Decreasing
$x<-\sqrt{6}$	+	+	+	incr
$-\sqrt{6}<x<-1$	+	-	-	decr
$-1<x<1$	-	-	+	incr
$1<x<\sqrt{6}$	+	-	-	decr
$x>\sqrt{6}$	+	+	+	incr

So $f(x)$ is increasing on the intervals $(-\infty,-\sqrt{6})$, $(-1,1)$ and $(\sqrt{6},\infty)$, and decreasing on the intervals $(-\sqrt{6},-1)$ and $(1,\sqrt{6})$.

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2. (d) Notice that the factor x^2+9 is always positive. The following table determines the intervals where $f''(x)$ is positive or negative.

Interval	$2x$	$(x^2-3)^3$	$f''(x)=\dfrac{2x(x^2+9)}{(x^2-3)^3}$	Concave Up or Down
$x<-\sqrt{3}$	-	+	-	down
$-\sqrt{3}<x<0$	-	-	+	up
$0<x<\sqrt{3}$	+	-	-	down
$x>\sqrt{3}$	+	+	+	up

So $f(x)$ is concave up on the intervals $(-\sqrt{3},0)$, and $(\sqrt{3},\infty)$, and concave down on the intervals $(-\infty,-\sqrt{3})$ and $(0,\sqrt{3})$.

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2. (e) It will be easier to evaluate the function $f(x)$ if we rewrite it as

$$f(x) = \frac{x^3+x^2-2x-3}{x^2-3} = \frac{x^3-2x}{x^2-3}+\frac{x^2-3}{x^2-3} = \frac{x(x^2-2)}{x^2-3}+1.$$

There are four critical values, $x=\pm 1$ and $x=\pm\sqrt{6}$.

$$f(1) = \frac{1\cdot(-1)}{-2}+1 = \frac{3}{2}; \qquad f(-1) = \frac{-1\cdot(-1)}{-2}+1 = \frac{1}{2};$$

$$f(\sqrt{6}) = \frac{\sqrt{6}\cdot 4}{3}+1 = 1+\tfrac{4}{3}\sqrt{6}; \qquad f(-\sqrt{6}) = \frac{-\sqrt{6}\cdot 4}{3}+1 = 1-\tfrac{4}{3}\sqrt{6}.$$

So there are four critical points, $(1,\tfrac{3}{2})$, $(-1,\tfrac{1}{2})$, $(\sqrt{6},1+\tfrac{4}{3}\sqrt{6})$ and $(-\sqrt{6},1-\tfrac{4}{3}\sqrt{6})$. A local minimum occurs when the function stops decreasing and starts increasing, which occurs at $(-1,\tfrac{1}{2})$ and $(\sqrt{6},1+\tfrac{4}{3}\sqrt{6})$, while a local maximum occurs when the function stops increasing and starts decreasing, which occurs at $(-\sqrt{6},1-\tfrac{4}{3}\sqrt{6})$ and $(1,\tfrac{3}{2})$.

There is an inflection point at $(0,f(0))=(0,1)$.

Local Minima: $(-1,\tfrac{1}{2})$, $(\sqrt{6},1+\tfrac{4}{3}\sqrt{6})$. Local Maxima: $(-\sqrt{6},1-\tfrac{4}{3}\sqrt{6})$, $(1,\tfrac{3}{2})$.

Inflection Point: $(0,1)$.

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2. (f) The function $f(x)=\dfrac{x^3+x^2-2x-3}{x^2-3}$ is undefined at $x=\pm\sqrt{3}$. There are vertical asymptotes at $x=\pm\sqrt{3}$, since

$$\lim_{x\to(-\sqrt{3})^-} f(x)=-\infty,\quad \lim_{x\to(-\sqrt{3})^+} f(x)=+\infty,\quad \lim_{x\to(\sqrt{3})^-} f(x)=-\infty, \text{ and } \lim_{x\to(\sqrt{3})^+} f(x)=+\infty.$$

Also, since

$$f(x) = \frac{x(x^2-2)}{x^2-3}+1 = x\cdot\frac{(x^2-3)+1}{x^2-3}+1 = x\cdot\left(\frac{x^2-3}{x^2-3}+\frac{1}{x^2-3}\right)+1$$

$$= x\cdot\left(1+\frac{1}{x^2-3}\right)+1 = \left(x+\frac{x}{x^2-3}\right)+1 = (x+1)+\frac{x}{x^2-3},$$

then,

$$\lim_{x\to\pm\infty}[f(x)-(x+1)] = \lim_{x\to\pm\infty}\frac{x}{x^2-3} = \lim_{x\to\pm\infty}\frac{\frac{x}{x}}{\frac{x^2}{x}-\frac{3}{x}} = \lim_{x\to\pm\infty}\frac{1}{x-\frac{3}{x}} = 0,$$

so $y=x+1$ is a slant asymptote.

Vertical Asymptotes: $x=\pm\sqrt{3}$. Slant Asymptote: $y=x+1$.

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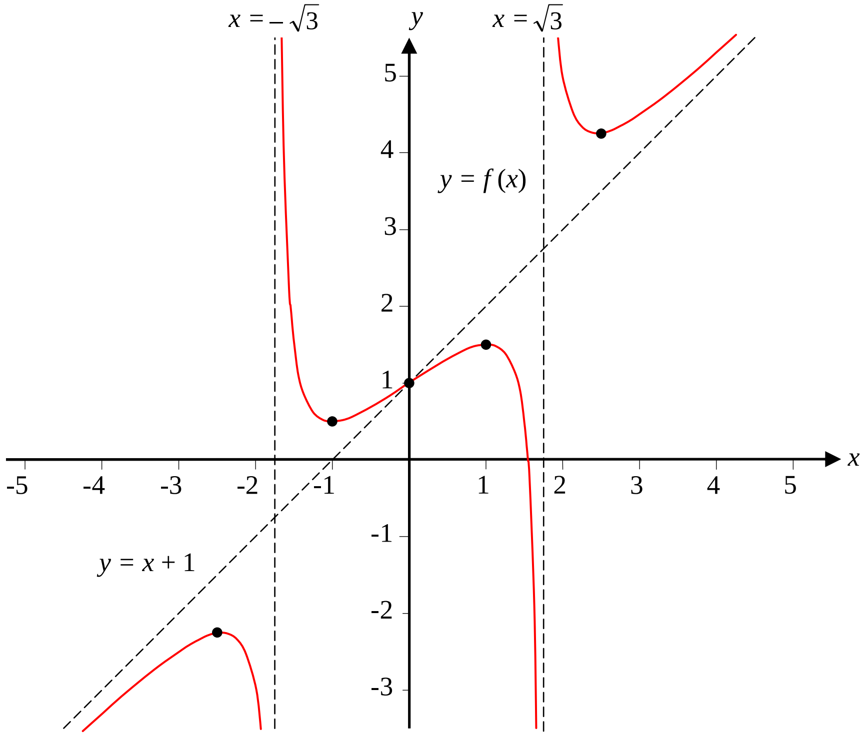
2. (g)



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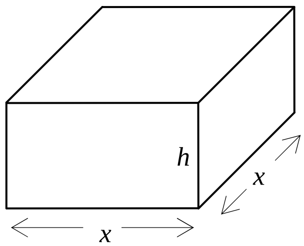
3. Let x be the length of the square base of the box and h its height. The volume is then $V = x^2 h = 16$ so that $h = \dfrac{16}{x^2}$. The cost of the base is \$40 per square metre, and the cost of the top is \$10 per square metre. The total cost of the box is then

$$C = 40x^2 + 4(20xh) + 10x^2 = 50x^2 + 80xh = 50x^2 + 80x \cdot \frac{16}{x^2}$$

$$= 50x^2 + \frac{1280}{x}.$$



So $$\frac{dC}{dx} = \frac{d}{dx}(50x^2 + 1280x^{-1}) = 100x - 1280x^{-2} = 100x - \frac{1280}{x^2}.$$

At a critical point $\dfrac{dC}{dx} = 0$ so $100x = \dfrac{1280}{x^2}$. Therefore $x^3 = \dfrac{1280}{100} = \dfrac{64}{5}$, so $x = \dfrac{4}{\sqrt[3]{5}}$.

The height is then $h = \dfrac{16}{x^2} = \dfrac{16}{x^3} \cdot x = \dfrac{16}{\frac{64}{5}} \cdot \dfrac{4}{\sqrt[3]{5}} = \dfrac{5}{\sqrt[3]{5}}$.

Notice that $\dfrac{d^2C}{dx^2} = 100 + 2560x^{-3} = 100x + \dfrac{2560}{x^3} > 0$, so this gives a minimum cost.

So the dimensions that minimize the cost are $x = \dfrac{4}{\sqrt[3]{5}}$ and $h = \dfrac{5}{\sqrt[3]{5}}$.

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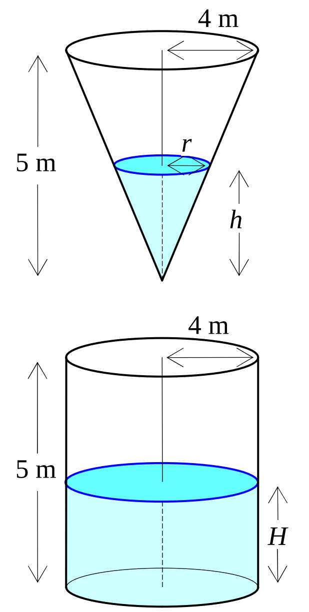
4. Let H represent the height of the water in the cylindrical tank, h the height of the cone of water in the conical tank and r its radius as shown in the diagram. By similar triangles, we know that

$$\frac{r}{h} = \frac{4}{5} \quad \Rightarrow \quad r = \tfrac{4}{5}h$$



The water originally filled the conical tank so the total volume of water is equal to the volume of the conical tank, i.e.

$$V = \tfrac{1}{3}\pi \cdot 4^2 \cdot 5 = \tfrac{80}{3}\pi \,.$$

The volume of water in the two tanks must add up to the total volume of $\tfrac{80}{3}\pi \text{ m}^3$, so

$$\tfrac{1}{3}\pi r^2 h + \pi \cdot 4^2 \cdot H = \tfrac{80}{3}\pi \quad \Rightarrow \quad \pi[\tfrac{1}{3}(\tfrac{4}{5}h)^2 h + 16H] = \tfrac{80}{3}\pi$$

or $$\tfrac{1}{3}(\tfrac{16}{25}h^2)h + 16H = \tfrac{80}{3} \quad \Rightarrow \quad \tfrac{16}{75}h^3 + 16H = \tfrac{80}{3}\,.$$

Multiplying by $\tfrac{75}{16}$ gives $h^3 + 75H = 125$. Therefore

$$\frac{d}{dt}(h^3 + 75H) = \frac{d}{dt}(125) \quad \Rightarrow \quad 3h^2\frac{dh}{dt} + 75\frac{dH}{dt} = 0\,.$$

Plugging in $h = 3$, $\dfrac{dh}{dt} = -0.5$ gives

$$3 \cdot 3^2 \cdot (-0.5) + 75\frac{dH}{dt} = 0 \quad \Rightarrow \quad 75\frac{dH}{dt} = \frac{27}{2} \quad \Rightarrow \quad \frac{dH}{dt} = \frac{27}{150} = \frac{9}{50}\,.$$

The water level in the cylindrical tank is rising at a rate of $\dfrac{9}{50} = 0.18 \text{ m/min}$.

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5. (a) The tangent line passes through the points $(p,q) = (8,0)$ and $(p,q) = (0,8)$ so its slope is

$$m = \frac{\Delta q}{\Delta p} = \frac{q_2 - q_1}{p_2 - p_1} = \frac{0-8}{8-0} = -1\,.$$

So the value of the derivative at the point $(p,q) = (2,6)$ is $\left.\dfrac{dq}{dp}\right|_{(2,6)} = -1$. Therefore

$$\varepsilon(2) = \frac{p}{q}\frac{dq}{dp} = \frac{2}{6}(-1) = -\frac{1}{3}\,.$$

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5. (b) Since the elasticity is between 0 and -1 ($-1 < \varepsilon < 0$), demand is inelastic, so revenue can be increased by <u>raising</u> the price from \$2.

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5. (c) $\dfrac{\Delta q}{q} \approx \varepsilon \dfrac{\Delta p}{p} \;\Rightarrow\; 0.05 \approx -\dfrac{1}{3}\cdot\dfrac{\Delta p}{p} \;\Rightarrow\; \dfrac{\Delta p}{p} \approx -3(0.05) = -0.15$.

Price must be lowered by approximately 15% to result in a 5% increase in demand.

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5. (d) Revenue is maximized when the elasticity is equal to -1 ($\varepsilon = -1$). Therefore

$$\frac{p}{q}\frac{dq}{dp} = -1 \;\Rightarrow\; \frac{dq}{dp} = -\frac{q}{p}.$$

Since $\dfrac{dq}{dp}$ is the slope of the tangent line T to the demand curve at (p,q) and $\dfrac{q}{p}$ is the slope of the line L connecting the origin to the point (p,q), these two lines must have opposite slopes and thus form equal angles with the p-axis. So the triangle formed by the lines L, T and the p-axis must form an isosceles triangle.

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6. (a) $f'(x) = \dfrac{1}{x}$

The linear approximation is given by

$$f(x) \approx f(a) + f'(a)(x-a) = f(1) + f'(1)(x-1),$$

or $\quad \ln x \approx \ln 1 + \tfrac{1}{1}(x-1) = 0 + 1(x-1) = x-1$.

Plugging in $x = 0.9$ gives

$$\ln(0.9) \approx 0.9 - 1 = -0.1.$$

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6. (b) $f''(x) = -x^{-2} = -\dfrac{1}{x^2}$

The number M is the maximum of the absolute value of the second derivative $f''(t)$ for t between x and a, i.e.,

$$M = \max_{x\le t\le a}\left| f''(t)\right| = \max_{0.9\le t\le 1}\left|-\frac{1}{t^2}\right| = \max_{0.9\le t\le 1}\left(\frac{1}{t^2}\right) = \frac{1}{0.9^2} = \frac{1}{0.81}.$$

(Since $1/t^2$ increases as t gets smaller, its maximum value will occur when t is as small as possible, which in this case is when $t = 0.9$). Therefore,

$$|E| \le \tfrac{M}{2}(x-a)^2 = \tfrac{1}{2}\cdot\tfrac{1}{0.81}(0.9-1)^2 = \tfrac{1}{1.62}(-0.1)^2 = \tfrac{0.01}{1.62} = \tfrac{1}{162} \approx 0.0062.$$

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6. (c) Since $f''(x) = -\dfrac{1}{x^2} < 0$, the graph of $f(x) = \ln x$ is concave down and so lies below its tangent line. Therefore the true value of $\ln(0.9)$ is actually less than the tangent line approximation so $\ln(0.9) < -0.1$. Since the error is less than $\frac{1}{162} \approx 0.0062$,

$$-0.1 - 0.0062 < \ln(0.9) < -0.1$$

so $\ln(0.9)$ lies in the interval $(-0.1062, -0.1000)$.

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6. (d) The quadratic approximation is given by

$$f(x) \approx f(a) + f'(a)(x-a) + \tfrac{1}{2} f''(a)(x-a)^2 = f(1) + f'(1)(x-1) + \tfrac{1}{2} f''(1)(x-1)^2$$

or $\quad \ln x \approx \ln 1 + \frac{1}{1}(x-1) + \frac{1}{2} \cdot (-\frac{1}{1^2})(x-1)^2 = 0 + (x-1) - \frac{1}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$.

Plugging in $x = 0.9$ gives

$$\ln(0.9) \approx (0.9-1) - \tfrac{1}{2}(0.9-1)^2 = -0.1 - \tfrac{1}{2}(-0.1)^2 = -0.1 - \tfrac{1}{2}(0.01) = -0.105.$$

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6. (e) The error bound for the linear approximation is $\frac{1}{2}\left| f''(t) \right| (0.1-1)^2$ where $0.9 < t < 1$, while the magnitude of the 2nd degree term of the quadratic approximation is $\frac{1}{2}\left| f''(1) \right| (0.1-1)^2$. If $0.9 < t < 1$, then

$$\left| f''(t) \right| = \frac{1}{t^2} > \frac{1}{1^2} = \left| f''(1) \right|,$$

so we expect the error bound $\frac{1}{2}\left| f''(t) \right| (0.1-1)^2$ to be greater than the magnitude of the 2nd degree term of the quadratic approximation $\frac{1}{2}\left| f''(1) \right| (0.1-1)^2$.
